1 The Circular Normal Distribution

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1.1 Introduction

Circular normally distributed quality characteristics appear commonly in machining and manufacturing operations. Any time there is an opportunity for the center line of a part or component to be displaced in two directions from a target position the circular normal distribution may govern the behavior of the displacement. Typical machining operations where circular normal data appear are turning, threading, milling of projections, drilling, boring, etc. Examples of situations which exhibit circular normal distributions are lengths of tubing that run out of alignment over their length, the positions of projections on large parts machined from the same stock, positions of drilled holes, etc. Another special case of circular normal data is the geographical position data reported by a global positioning system (GPS) receiver.

An example of a problem involving circular normal data is shown in Figure 1. The Figure shows the (x, y) positions of parts and their intended target at the origin of the (x, y) coordinate system. Individually the distributions of x and y are normal, but it is the combined effect of x and y that is of concern. The relevant displacement of a part from its intended position is measured by the radial distance r taken relative to $(x, y) = (0, 0)$. The value of r for a part depends on its x and y displacements according to:

$$r = \sqrt{x^2 + y^2} \tag{1}$$

It is the distribution of r that is circular normal. In some cases (x, y) data may be available and in others only the r values might be measured. A typical question to be addressed that involves the circular normal distribution for Figure 1 would be to determine an upper spec limit for r such that a specified fraction of the observations meet the spec.

The purpose of this document is to present the characteristics and behavior of the circular normal distribution and outline the steps involved in analyzing circular normal data.

1.2 Probability Density Function

The probability density function for the general case of a bivariate normal distribution is given by:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} \tag{2}$$

where ρ is the correlation coefficient between x and y. In the special case of a circular normal distribution we have $\sigma_x = \sigma_y$ and $\rho = 0$. Then the probability density function reduces to:

$$f(x, y) = \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_x}\right)^2\right]} \tag{3}$$

Suppose that we further simplify the problem by taking the origin for the (x, y) system to be at $(\mu_x, \mu_y) = (0, 0)$. (This is easily achieved in practice by replacing the x and y values with the transformations $x' = x - \mu_x$ and $y' = y - \mu_y$. This transformation is called mean adjusting the

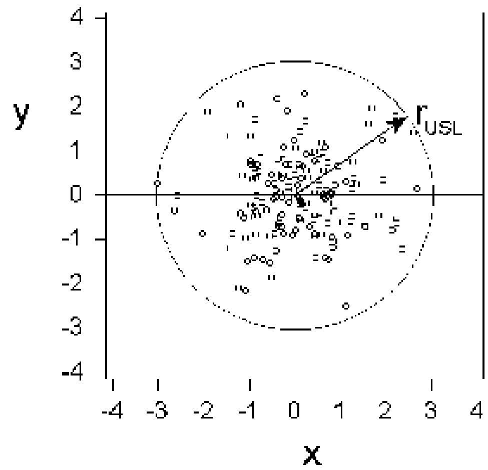

Figure 1: Example of Circular Normal Data

data.) Then we can rewrite the probability density function $f(x, y)$ in terms of a polar coordinate system (r, θ) with its origin at $(x, y) = (0, 0)$. Then the point (x, y) will have polar coordinates:

$$r = \sqrt{x^2 + y^2} \tag{4}$$

and:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{5}$$

The probability density function $f(x, y)$ becomes:

$$f(r) = \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} \tag{6}$$

The polar angle θ does not affect the density function because $f(x, y)$ depends only on how far a point falls from the origin and not in any way on the direction.

Figure 2 shows a three dimensional representation of the circular normal probability density function $f(x, y)$. The height of the surface in Figure 2 indicates the magnitude of $f(x, y)$ or $f(r)$. The distribution is centered at $(x, y) = (0, 0)$ and the standard deviations are $\sigma_x = \sigma_y = 1$. Figure 3 shows the corresponding distributions of x and y. Notice from both of the Figures that the standard deviations in the x and y directions are equal and that the x and y directions are independent of each other.

The probability density of r as given by Equation 4 is shown in Figure 4. This curve is normal in shape and corresponds to any vertical slice taken through the origin of Figure 2 such as in Figure 3. These Figures were all constructed with $\sigma_x = \sigma_y = 1$.

The probability of finding an observation in the interval from x to $x + dx$ and from y to $y + dy$ is given by $f(x, y)\, dxdy$. Similarly, the probability of finding an observation in the interval from r to $r + dr$ is given by $2\pi r f(r)\, dr$. Since we will generally be interested in the distribution of r rather than just either x or y the distribution of r deserves special consideration. The distribution of $2\pi r f(r)$ vs. r is shown in Figure 5. (This Figure used $\sigma_x = \sigma_y = 1$.) Histograms of r data for real processes that follow the circular normal distribution should look like Figure 5. The mean and the standard deviation of the distribution of r are indicated in the Figure and the methods for calculating them are presented below.

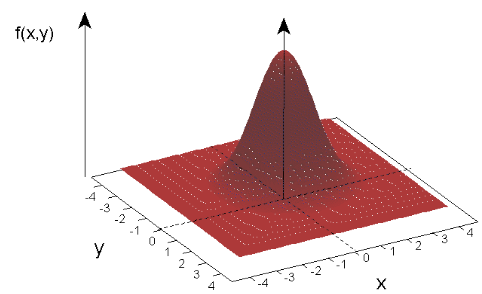

Figure 2: Circular Normal Probability Density Function $f(x, y)$

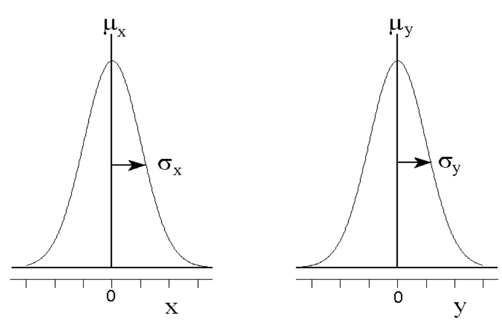

Figure 3: Normal Distributions of x and y

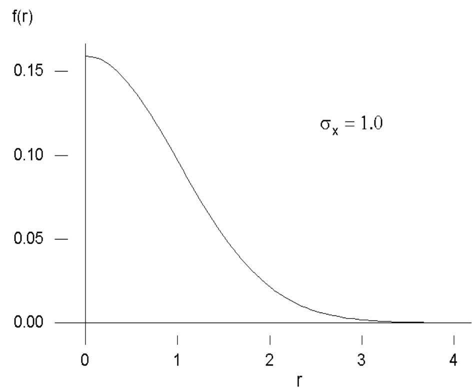

Figure 4: Circular Normal Probability Density Function $f(r)$

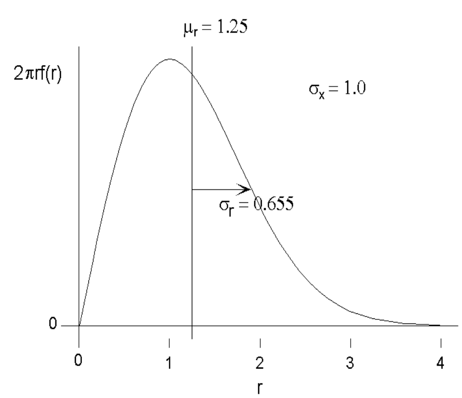

Figure 5: Probability of Finding an Observation Between r and $r + dr$

1.3 Normalization

The circular normal distribution must be normalized, that is, the total probability for $0 \leq r < \infty$ must be unity. The normalization integral is:

$$\begin{aligned} P\left(0 \leq r \leq \infty\right) &= \int_0^\infty f\left(r\right) 2\pi r dr \\ &= \int_0^\infty \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} 2\pi r dr \\ &= 2\int_0^\infty e^{-\left(\frac{r}{\sqrt{2}\sigma_x}\right)^2} \left(\frac{r}{\sqrt{2}\sigma_x}\right)\left(\frac{dr}{\sqrt{2}\sigma_x}\right) \\ &= 2\int_0^\infty e^{-u^2} u du \\ &= 2\left(\frac{1}{2}\right) \\ &= 1 \end{aligned} \tag{7}$$

where we made the substitution $u = \frac{r}{\sqrt{2}\sigma_x}$. This just confirms that volumes under the surface $f\left(x, y\right)$ can be interpreted as the probability of finding an observation within specified ranges of x and y.

1.4 The Mean of r

In general, the mean of the probability distribution of $f\left(q\right)$ is given by the expectation value:

$$\mu = E\left(q\right) = \int\limits_{-\infty}^{\infty} q f\left(q\right) dq \tag{8}$$

where $-\infty < q < \infty$. The mean of the circular normal distribution is then:

$$\begin{aligned} \mu_r &= E\left(r\right) \\ &= \int_0^\infty r f\left(r\right) 2\pi r dr \\ &= \int_0^\infty \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} 2\pi r^2 dr \\ &= \int_0^\infty \frac{1}{\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} r^2 dr \\ &= 2\sqrt{2}\sigma_x \int_0^\infty e^{-\left(\frac{r}{\sqrt{2}\sigma_x}\right)^2} \left(\frac{r}{\sqrt{2}\sigma_x}\right)^2 \left(\frac{dr}{\sqrt{2}\sigma_x}\right) \\ &= 2\sqrt{2}\sigma_x \int_0^\infty e^{-u^2} u^2 du \\ &= 2\sqrt{2}\sigma_x \left(\frac{\sqrt{\pi}}{4}\right) \\ &= \sqrt{\frac{\pi}{2}}\sigma_x \\ &= 1.25\sigma_x \end{aligned} \tag{9}$$

where we made the substitution $u = \frac{r}{\sqrt{2}\sigma_x}$.

1.5 The Variance of r

In general, the variance of the probability distribution of q is given by:

$$\begin{aligned} \sigma^2 &= E\left(q^2\right) - \left(E\left(q\right)\right)^2 \\ &= \int_{-\infty}^\infty q^2 f\left(q\right) dq - \mu^2 \end{aligned} \tag{10}$$

The value of $E\left(r^2\right)$ for the circular normal distribution is then:

$$\begin{aligned} E\left(r^2\right) &= \int_0^\infty r^2 f\left(r\right) 2\pi r dr \\ &= 2\pi \int_0^\infty f\left(r\right) r^3 dr \\ &= 2\pi \int_0^\infty \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} r^3 dr \\ &= 4\sigma_x^2 \int_0^\infty e^{-\left(\frac{r}{\sqrt{2}\sigma_x}\right)^2} \left(\frac{r}{\sqrt{2}\sigma_x}\right)^3 \left(\frac{dr}{\sqrt{2}\sigma_x}\right) \\ &= 4\sigma_x^2 \int_0^\infty e^{-u^2} u^3 du \\ &= 4\sigma_x^2 \left(\frac{1}{2}\right) \\ &= 2\sigma_x^2 \end{aligned} \tag{11}$$

where we made the substitution $u = \frac{r}{\sqrt{2}\sigma_x}$. Then the value of the variance of r is:

$$\begin{aligned} \sigma_r^2 &= E\left(q^2\right) - \left(E\left(q\right)\right)^2 \\ &= 2\sigma_x^2 - \left(\sqrt{\frac{\pi}{2}}\sigma_x\right)^2 \\ &= 2\sigma_x^2 - \left(\frac{\pi}{2}\right)\sigma_x^2 \\ &= \left(2 - \frac{\pi}{2}\right)\sigma_x^2 \\ &= 0.429\sigma_x^2 \end{aligned} \tag{12}$$

and the standard deviation of r is:

$$\sigma_r = \sqrt{2 - \frac{\pi}{2}}\sigma_x = 0.655\sigma_x \tag{13}$$

1.6 Direct Evaluation of σ_r

Let the vector $\vec{r}$ be bivariate normally distributed and a function of x and y with $\mu_x = \mu_y = 0$ and $\sigma_x = \sigma_y$. That is, the origin for the x and y axes lies at the center of the bivariate normal distribution and $\vec{r}$ is taken from the origin to the point with coordinates (x, y). We wish to characterize the distribution of $r = |\vec{r}|$. If the ith point has coordinates (x_i, y_i) then:

$$r_i = \sqrt{x_i^2 + y_i^2} \tag{14}$$

A simple model for the r_i values is:

$$r_i = \mu_r + \epsilon_i \tag{15}$$

so

$$\epsilon_i = r_i - \mu_r \tag{16}$$

The variance of the r_i is given explicitly by:

$$\begin{aligned} \sigma_r^2 &= \frac{1}{n}\sum \epsilon_i^2 \\ &= \frac{1}{n}\sum \left(r_i - \mu_r\right)^2 \\ &= \frac{1}{n}\sum \left(r_i^2 - 2r_i\mu_r + \mu_r^2\right) \\ &= \frac{1}{n}\sum r_i^2 - \frac{2\mu_r}{n}\sum r_i + \frac{1}{n}\sum \mu_r^2 \\ &= \frac{1}{n}\sum r_i^2 - 2\mu_r\frac{\sum r_i}{n} + \frac{1}{n}(n\mu_r^2) \\ &= \frac{1}{n}\sum r_i^2 - 2\mu_r^2 + \mu_r^2 \\ &= \frac{1}{n}\sum r_i^2 - \mu_r^2 \\ &= \frac{1}{n}\sum \left(x_i^2 + y_i^2\right) - \mu_r^2 \\ &= \frac{1}{n}\sum x_i^2 + \frac{1}{n}\sum y_i^2 - \mu_r^2 \\ &= \sigma_x^2 + \sigma_y^2 - \mu_r^2 \\ &= 2\sigma_x^2 - \mu_r^2 \end{aligned} \tag{17}$$

This condition is consistent with the values $\sigma_r = \sqrt{2 - \frac{\pi}{2}}\sigma_x$ and $\mu_r = \sqrt{\frac{\pi}{2}}\sigma_x$ determined above.

1.7 Estimation of σ_x

An accurate and precise estimate of σ_x is necessary to use the circular normal distribution to make predictions. When the data take the form of the raw (x, y) displacements then σ_x can be calculated directly. Since the circular normal distribution requires $\sigma_x = \sigma_y$ then either the x data or the y data can be used to make the necessary estimate. A better estimate can be obtained by combining the x and y values and calculating the resultant standard deviation. Since there will be $2n$ observations that contribute to the calculation of σ_x from n points, the value obtained by combining the observations will have less variation by a factor of $\sqrt{2}$ compared to an estimate of σ_x obtained from just the x or just the y values.

When the (x, y) data are not known and only their r values are available alternative techniques must be used to estimate σ_x. Equations 9 and 13 indicate that both μ_r and σ_r are proportional to σ_x. These equations can be solved to obtain the desired estimate:

$$\sigma_x = \sqrt{\frac{2}{\pi}}\mu_r \simeq \sqrt{\frac{2}{\pi}}\bar{r} \tag{18}$$

and

$$\sigma_x = \frac{\sigma_r}{\sqrt{2 - \frac{\pi}{2}}} \simeq \frac{s_r}{\sqrt{2 - \frac{\pi}{2}}} \tag{19}$$

where $\bar{r}$ is the mean r value and s_r is the standard deviation of the r values. These two estimates do not have the same precision. s_r has more variability as an estimator of σ_x compared to the estimate provided by $\bar{r}$ so Equation 18 is preferred.

Note that since μ_r and σ_r are both proportional to σ_x their ratio must be a constant:

$$\begin{aligned} \frac{\mu_r}{\sigma_r} &= \frac{\sqrt{\frac{\pi}{2}}\sigma_x}{\sqrt{2-\frac{\pi}{2}}\sigma_x} \\ &= \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2-\frac{\pi}{2}}} \\ &= \sqrt{\frac{\pi}{4-\pi}} \\ &= 1.913 \end{aligned} \tag{20}$$

Estimates of the ratio μ_r/σ_r from sample data will be noisy and will deviate from the constant 1.913. Figure 6 shows approximate 95% sampling bounds for μ_r/σ_r. (These bounds were determined by simulation.) This relationship will be made use of later to test if sample data might come from a circular normal population.

1.8 Circular Normal Probabilities

In polar coordinates the circular normal probability density function is given by:

$$f(r) = \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} \tag{21}$$

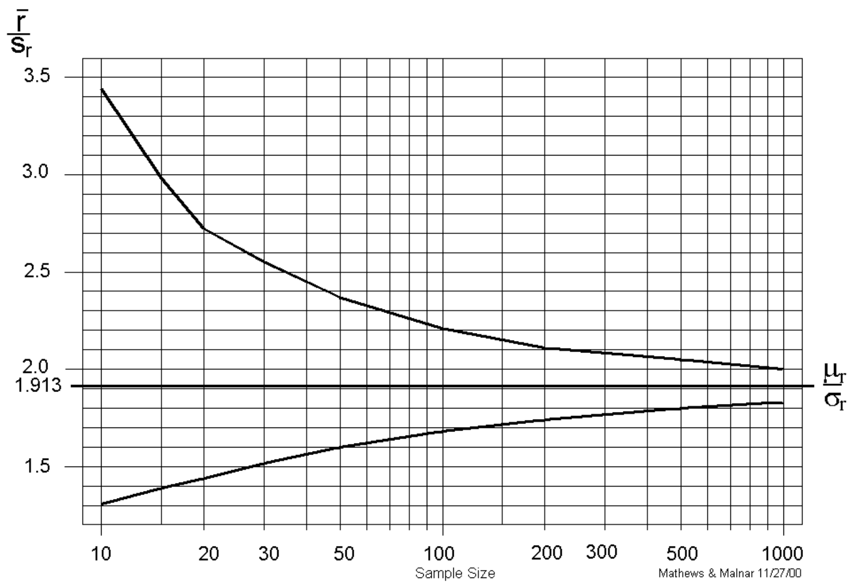

Figure 6: 95% Sampling Bounds for μ_r/σ_r as a Function of Sample Size

The probability of finding an observation within a distance r of the origin is:

$$
\begin{aligned}
P\left(0 \le r' \le r; \sigma_x\right) &= \int_0^r f\left(r'\right) 2\pi r' dr' \\
&= \int_0^r \frac{1}{2\pi\sigma_x^2} e^{-\frac{1}{2}\left(\frac{r'}{\sigma_x}\right)^2} 2\pi r' dr' \\
&= 2\int_0^r e^{-\left(\frac{r'}{\sqrt{2}\sigma_x}\right)^2} \left(\frac{r'}{\sqrt{2}\sigma_x}\right)\left(\frac{dr'}{\sqrt{2}\sigma_x}\right) \\
&= 2\int_0^r e^{-u^2} u du \\
&= 2\left[-\frac{1}{2}e^{-u^2}\right]_{u=0}^{u=\frac{r}{\sqrt{2}\sigma_x}} \\
&= 1 - e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2}
\end{aligned}
\tag{22}
$$

where we made the substitution $u = \frac{r'}{\sqrt{2}\sigma_x}$. The area of integration is shown in Figure 7 and the resulting probability function is plotted in Figure 8.

1.9 Special Circular Normal Probabilities

The previous section showed that the probability of finding an observation within a distance r of the origin is:

$$
\begin{aligned}
P\left(0 < r' < r; \sigma_x\right) &= \int_0^a f\left(r'\right) 2\pi r' dr' \\
&= 1 - e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2}
\end{aligned}
\tag{23}
$$

If observations with $0 \le r' \le r$ are considered 'good' then the ones with $r' > r$ are 'bad'. If the fraction of the observations that are bad is p (i.e. the fraction defective) then:

$$1 - p = 1 - e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} \tag{24}$$

or

$$p = e^{-\frac{1}{2}\left(\frac{r}{\sigma_x}\right)^2} \tag{25}$$

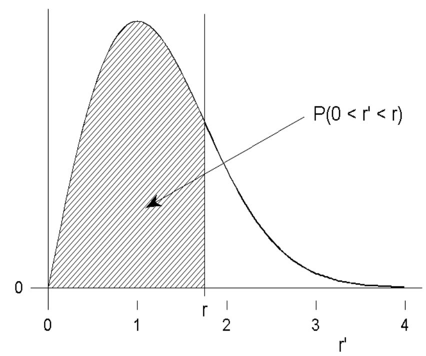

Figure 7: Area of Integration for $P\left(0 < r' < r\right)$

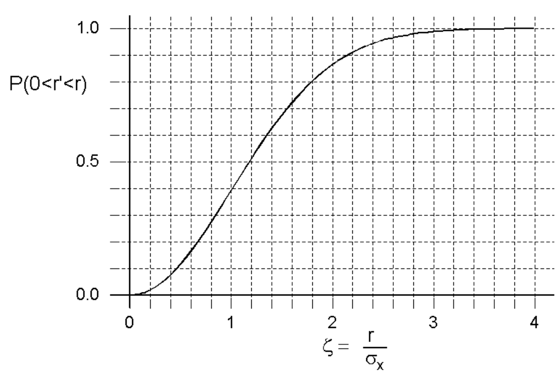

Figure 8: Probability of an Observation Falling Within Distance r of the Origin $P(0 \leq r' \leq r)$

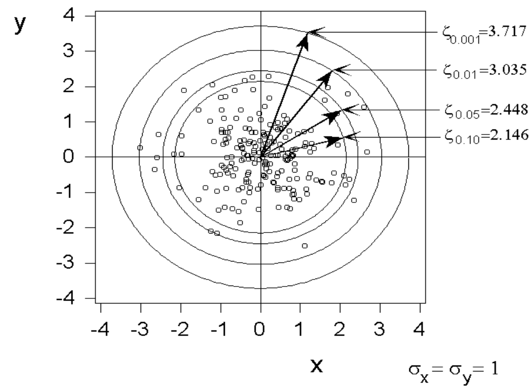

Figure 9: Special Values of $\zeta_p = r/\sigma_x$

The quantity $\frac{r}{\sigma_x}$ is equivalent to the standard units or z values from the univariate normal distribution. To avoid confusing univariate normal and circular normal standard units we can introduce the notation $\zeta = \frac{r}{\sigma_x}$. ($\zeta$ is the Greek letter zeta.) The values of the fraction good and the fraction defective for special values of $\zeta = \frac{r}{\sigma_x}$ are:

ζ	$(1-p)$	p
1.0	0.3935	0.6065
2.0	0.8647	0.1353
3.0	0.9889	0.0111
4.0	0.99966	0.00034

For example, the table indicates that 98.89% of a circular normal population falls within $\zeta = 3$ standard deviations of the target at $r = 0$.

Equation 25 can be easily solved for $\zeta = \frac{r}{\sigma_x}$:

$$\zeta = \frac{r}{\sigma_x} = \sqrt{-2\ln(p)} \tag{26}$$

Some special values of ζ are:

p	ζ_p
0.10	2.146
0.05	2.448
0.01	3.035
0.001	3.717

where the subscript p indicates the right tail area under the circular normal distribution just like z_p is the normal distribution z value with right tail area p. For example, we can write $\zeta_{0.01} = 3.035$. These special values are shown in Figure 9.

1.10 Circular Normal Probability Plots

A histogram of circular normal data should look like Figure 5 but a more sensitive graphical test for the circular normal condition is desired. What we need is a circular normal probability plot analogous to the normal probability plot for univariate normal data. Generating a circular normal probability plot is actually easier than generating a normal probability plot because special probability paper is not required and the necessary calculations can be performed in a simple spreadsheet. The procedure is indicated below:

1. Collect the (x, y) or r data. If (x, y) data are used calculate their r values. Let n be the number of observations.

2. Sort the r values from smallest to largest. Let the smallest r value be r_1, the next smallest r_2, ..., and the largest r value r_n.

3. Calculate the midband percentile corresponding to each r_i value. The midband percentile for the ith observation is given by:
$$p_i = \frac{i - \frac{1}{2}}{n} \tag{27}$$

4. Find the ζ value that corresponds to each midband percentile from:
$$\zeta_i = \sqrt{-2 \ln (p_i)} \tag{28}$$

5. Plot r_i vs. ζ_i for all n points from $i = 1$ to n.

6. Estimate σ_x from the data. Draw a line from the origin at $(\zeta, r) = (0, 0)$ through the point $(\zeta, r) = (3, 3\sigma_x)$. ($\zeta = 3$ was an arbitrary choice. There's nothing sacred about $\zeta = 3$.)

7. Inspect the plotted points to see if they fall along the line. If the points substantially fall along the line conclude that the distribution of r is circular normal. If the points do not follow the line conclude that the distribution of r is not circular normal. Apply the usual guidelines for interpreting probability plots.

Figure 10 shows a normal probability plot for an 80 point sample taken from a circular normal population. The tendency for the points to generate downward curvature is characteristic of circular normal data on normal paper. Figure 11 shows the same data plotted on a circular normal probability plot. The points fall along an approximately straight line indicating that they probably come from a circular normal population. Both plots were generated in Minitab. The normal plot was generated with Minitab's *%normplot* macro provided in Minitab. The circular normal plot was generated with a custom macro called *%circnorm* written by the authors. *%circnorm* is just an implementation of the seven step procedure given above.

One of the important advantages of circular normal plots is that they are sensitive to shifts in the centering of the process. Consider the circular normal (x, y) data plotted in Figure 1. These data are centered at $(x, y) = (0, 0)$ as they're supposed to be and they are circular normally distributed so their r values would plot up as an approximately straight line in a circular normal probability plot. However, suppose that the whole data set shifted in the x or y direction to a new center position given by (μ_x, μ_y). A circular normal plot of the r values with r taken from the original origin at $(0, 0)$ will no longer show a straight line. That is, the circular normal plot 'detects' that something is wrong with the data set. The shifted data are still circular normal, but now they're circular normal relative to the new origin. A circular normal probability plot of r values taken relative to (μ_x, μ_y) will once again show a straight line.

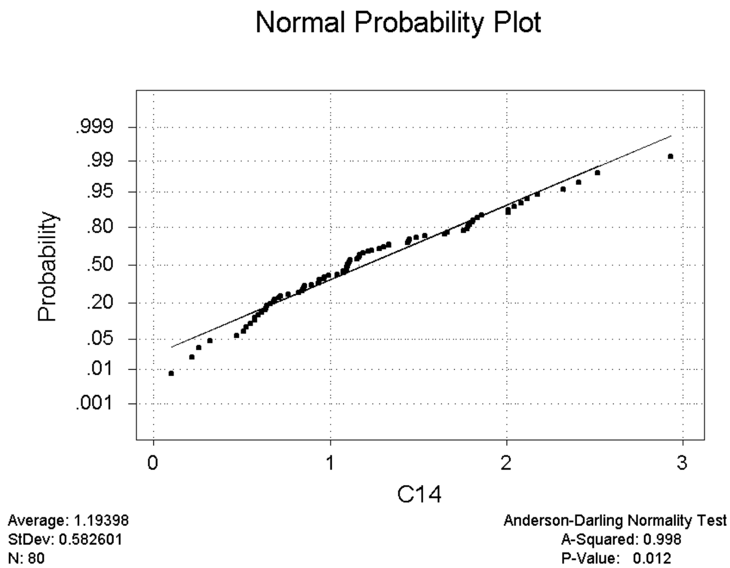

Figure 10: Normal Probability Plot of Circular Normal Data

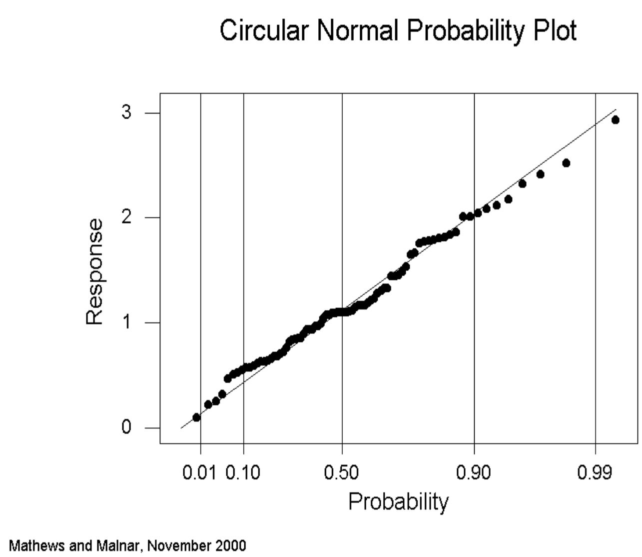

Figure 11: Circular Normal Probability Plot of Circular Normal Data

1.11 Consequences of Mishandling Circular Normal Data

Many quality practitioners mistakenly treat circular normal data as if they were univariate normal. This leads to incorrect spec limits, estimates for fractions defective, and control limits that increase the consumer's risk of receiving defective product.

As an example, consider the problem of setting a one sided upper spec limit on the distribution shown in Figure 5. If the data are treated as if they were univariate normal instead of circular normal then we would calculate estimates for μ_r and σ_r for the distribution parameters. The erroneous one sided upper spec limit that is expected to deliver fraction defective $p = 0.001$ is given by:

$$\begin{aligned} r_{USL} &= \mu_r + z_{0.001}\sigma_r \\ &= \mu_r + 3.09\sigma_r \end{aligned} \tag{29}$$

In fact, the circular normal nature of r actually makes the fraction defective much larger than $p = 0.001$. The value of the upper spec limit given by Equation 29 in the context of the circular normal distribution corresponds to:

$$\begin{aligned} r_{USL} &= \mu_r + 3.09\sigma_r \\ &= \sqrt{\tfrac{\pi}{2}}\sigma_x + 3.09\sqrt{2 - \tfrac{\pi}{2}}\sigma_x \\ &= 3.28\sigma_x \end{aligned} \tag{30}$$

With $\zeta = 3.28$ Equation 25 gives us:

$$\begin{aligned} p &= e^{-\frac{1}{2}\zeta^2} \\ &= e^{-\frac{1}{2}(3.28)^2} \\ & 0.0046 \end{aligned} \tag{31}$$

This indicates that use of the upper spec limit from Equation 29, which was intended to deliver a fraction defective of just $p = 0.001$, results in a defective rate 4.6 times larger! The correct upper spec limit is given using $\zeta_{0.001} = 3.717$ instead of $\zeta = 3.28$. The resulting value for r_{USL} is given by:

$$\begin{aligned} r_{USL} &= \zeta_p\sigma_x \\ &= \zeta_{0.001}\sigma_x \\ &= 3.72\sigma_x \end{aligned} \tag{32}$$

If the (x, y) data to calculate σ_x are not available directly then the estimate of μ_r should be used instead:

$$\begin{aligned} r_{USL} &= \zeta_{0.001}\sigma_x \\ &= 3.72\sqrt{\tfrac{2}{\pi}}\mu_r \\ &= 2.97\mu_r \end{aligned} \tag{33}$$

This provides a good benchmark for working with circular normal data: the upper spec limit on r must be 3 times the mean of r to obtain a fraction defective of 0.1% or:

$$P\left(0 < r < 3\mu_r\right) = 0.999 \tag{34}$$

When σ_x is not known but μ_r is known Equation 33 can be generalized for other values of p:

$$\begin{aligned} r_{USL} &= \zeta_p\sigma_x \\ &= \sqrt{-2\ln(p)}\sqrt{\tfrac{2}{\pi}}\mu_r \\ &= \tfrac{2}{\sqrt{\pi}}\sqrt{-\ln(p)}\mu_r \\ &= 1.128\sqrt{-\ln(p)}\mu_r \end{aligned} \tag{35}$$

1.12 Procedure for Analyzing Circular Normal Data

The following procedure outlines the steps that should be followed to analyze circular normal data. The procedure follows four basic steps: data collection, circular normal distribution validation, estimation of the single distribution parameter σ_x, and calculation of a fraction defective or a specification limit.

Recognize that circular normal data are collected in two common forms. In some cases, such as when the orientation of the part being inspected has some consistently defined orientation, the x and y positions of the part may be available. In other cases where the orientation of the part is arbitrary only maximum run-out or r values can be measured. In either case the data may be distributed according to the circular normal distribution but the first steps of the analysis process are different for the two types of data.

1. Collect the (x, y) or r data.

2. If only the r data are available skip to step 3. If the (x, y) data are available:

   (a) Construct normal probability plots of x and y and confirm that their distributions are normal.

   (b) Calculate the sample standard deviations s_x and s_y and use an F test to confirm that $\sigma_x = \sigma_y$. If this condition is not met do not proceed.

   (c) Confirm that x and y are independent of each other by using a scatter plot of y vs. x or the correlation coefficient r^2. The scatter plot should show a cloud of points with no obvious predictive relationship between x and y. The correlation coefficient r^2 should be close to 0. If the independence condition is not met do not proceed.

   (d) Calculate the sample means $\bar{x}$ and $\bar{y}$. Use one sample t tests to confirm that μ_x and μ_y are indistinguishable from zero. If one or both of them are not then the distribution is biased from the origin of the (x, y) measurement system. In this case mean adjust the x and y values using the transformations:

   $$x_i' = x_i - \bar{x} \tag{36}$$

   and

   $$y_i' = y_i - \bar{y} \tag{37}$$

   (e) Calculate the r_i from Equation 4 using the (x_i, y_i) if $\mu_x = \mu_y = 0$ or from the (x_i', y_i') if $\mu_x \neq 0$ and/or $\mu_y \neq 0$.

3. Construct the histogram of r values and confirm that it looks like Figure 5. Construct the circular normal probability plot of the r values and confirm that the data fall on an approximately straight line. If possible, use the Kolmogorov test to confirm that the distribution of r follows the circular normal distribution. If the histogram does not look like the Figure, or if the points on the circular normal plot deviate substantially from a straight line, or if the Kolmogorov test indicates that the circular normal model does not fit the data then do not proceed.

4. Estimate μ_r with $\bar{r}$ and σ_r with s_r. Confirm that $\mu_r/\sigma_r \simeq \bar{r}/s_r \simeq 1.91$. Compare the value you obtain to the 95% confidence limits taken from Figure 6. If your value of $\bar{r}/s_r$ falls outside this interval do not proceed. (This test is somewhat redundant and can be skipped if the data passes the tests in Figure 3.)

5. If the (x, y) values are known combine them to estimate σ_x. If only the r values are known estimate σ_x from Equation 18.

6. If the fraction defective is required for a specified upper spec limit on r use:

$$p = e^{-\frac{1}{2}\left(\frac{r_{USL}}{\sigma_x}\right)^2} \tag{38}$$

7. If an upper spec limit on r is required for a specified value of p use:

$$r_{USL} = \zeta_p \sigma_x \tag{39}$$

1.13 Control Charts for Circular Normal Data

We wish to identify an appropriate SPC control chart and the required chart design factors that can be used for statistics calculated from sample data taken from a circular normal population. Generic Shewhart control chart limits are:

$$\begin{aligned} CL_w &= \mu_w \\ (UCL/LCL)_w &= \mu_w \pm 3\sigma_w \end{aligned} \tag{40}$$

where w is a statistic calculated from sample data. These limits are valid when the distribution of w is normal or at least approximately normal. When the distribution of the test statistic deviates from normality the usual run rules and overall performance of the control chart deviates from intentions. This is not to say that a chart for a non-normally distributed statistic is not useful, just that the usual rules don't apply and special care has to be taken by the SPC practitioner. The alternative is to use large sample sizes which are cost prohibitive.

Two characteristics of the circular normal distribution force us to consider special treatment in the construction and operation of SPC charts. These conditions are:

1. The mean and standard deviation of the circular normal distribution are not independent.

2. The circular normal distribution deviates substantially from normality. (See Figure 12.)

The first condition works to our advantage - it means that it is only necessary to keep one control chart for circular normal data because one chart reflects both location and variation. (This is analogous to np and c charts.) The location statistic for circular normal data is the mean of the r_i given by $\bar{r}$. The relevant variation statistics are the sample standard deviation s_r or the sample range R_r. A control chart for any one of these statistics: $\bar{r}$, s_r, or R_r will work, however, s_r and R_r are noisy statistics compared to $\bar{r}$. This means that the preferred control chart for circular normal data is the chart of sample means, i.e. the $\bar{r}$ chart.

If μ_r is known, the center line for the $\bar{r}$ chart is given by:

$$CL_{\bar{r}} = \mu_r \tag{41}$$

and the upper and lower control limits are:

$$\begin{aligned} (UCL/LCL)_{\bar{r}} &= \mu_r \pm 3\sigma_r/\sqrt{n} \\ &= \mu_r \pm \frac{3}{\sqrt{n}}\left(\mu_r\sqrt{\frac{4-\pi}{\pi}}\right) \\ &= \mu_r\left(1 \pm 3\sqrt{\frac{4-\pi}{\pi n}}\right) \end{aligned} \tag{42}$$

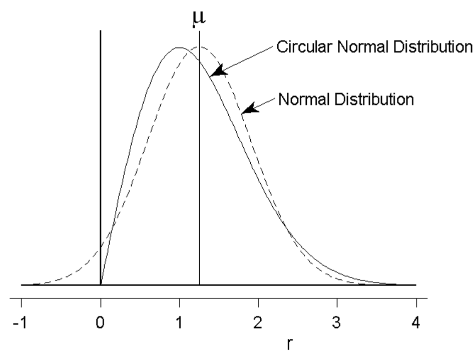

Figure 12: Normal and Circular Normal Probability Distribution Functions

where the factor $1/\sqrt{n}$ is the usual sampling theory contraction factor for means and Equation 13 was used to replace σ_r. When μ_r is not known it should be approximated with $\bar{\bar{r}}$ determined from at least 150 total observations, for example from 30 subgroups of size $n = 5$ or from 20 subgroups of size $n = 8$.

The second condition above, the nonnormality of the circular normal distribution, forces us to consider sample sizes for $\bar{r}$ charts that are larger than charts generally kept for normally distributed statistics. As with all distributions, sample means from a circular normal population become approximately normally distributed as the sample size gets large. However, the circular normal distribution is very asymmetric and this behavior persists in means of samples of size $n = 5$ and 8 which are upper limits for practical sizes for most applications. This means there are two choices: we either acknowledge that the distribution of $\bar{r}$ is not sufficiently normal for small samples and commit to large sample sizes or we devise some special guidelines for small sample sizes. Since money always wins we will go with the second choice. This means that it will be essential for the SPC practitioner to stay conscious of the deviations of the $\bar{r}$ chart from the usual expected behavior of charts for location where the distribution of the statistic is normal.

Figure 12 shows the circular normal distribution of r and the severity of its deviation from normality. The long right tail and the short left tail persist in $\bar{r}$ values taken from small samples like $n = 5$ and 8. Given UCL_r from Equation 42 the long right tail means that there will be a greater than usual tendency for circular normal data to generate Type 1 errors on the top half of the chart. (Recall that a Type 1 error occurs when a false out-of-control signal is received.) And given the LCL_r from Equation 42 the short left tail means that there will be a lesser tendency for circular normal data to generate Type 1 errors from the bottom half of the chart. These behaviors will make the $\bar{r}$ chart with these control limits more sensitive to shifts to higher values of μ_r and less sensitive to shifts to lower values of μ_r. This means that Type 2 error rates will be lower than usual for the top half of the chart and higher than usual for the bottom half of the chart. (Recall that a Type 2 error occurs when a false in-control signal is received.) As long as these qualifications are understood by the SPC practitioner and he stays conscious of the $\bar{r}$ chart's unusual behaviors the $\bar{r}$ chart for circular normal data can be managed safely.

Figure 13 shows an example $\bar{r}$ chart. Samples of size $n = 5$ were drawn from a circular normal population with $\sigma_x = 1.0$ (and by implication $\mu_r = 1.251$). The control limits were calculated

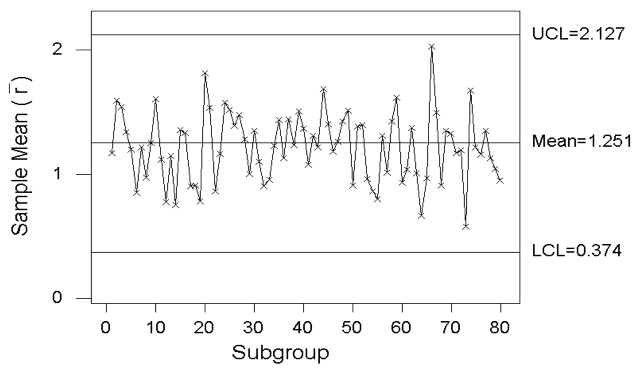

Figure 13: SPC Chart for $\bar{r}$

from Equation 42 using $\mu_r = 1.251$ but the limits determined from $\bar{\bar{r}}$ are nearly indistinguishable. The chart was created in Minitab and all 8 of Minitab's variables chart run rules were active but none of them turned on in this limited example. Because of the higher than usual Type 1 error rate on the top half of the chart we shouldn't have been surprised if some out-of-control signals were found there.

The $\bar{r}$ chart for circular normal data is sensitive to any change in the parameters of the distribution. This chart can detect increases in the standard deviation σ_x, decreases in σ_x, and shifts of the circular normal distribution from its intended target position. Figure 14 shows an example $\bar{r}$ chart with 4 distinct regions indicated by A, B, C, and D. Each region contains 20 subgroups and each plotted $\bar{r}$ value was calculated from a sample of size $n = 5$ taken from a circular normal distribution. The control limits were calculated for $\sigma_x = 1$.

- The data in region A were taken under the same conditions as those shown in Figure 13 ($\sigma_x = 1$). As expected, all of the points in region A fall within the control limits and there are no out-of-control patterns present.

- The points in region B were sampled from a circular normal distribution with $\sigma_x = 2$. The increased spread in the (x, y) points about the target at $(0, 0)$ results in an increase in the $\bar{r}$ value that is detected by the control chart. The chart clearly shows that process B is out of control.

- The points in region C were sampled from a circular normal distribution with $\sigma_x = 0.5$. The tighter spread in the (x, y) points about the target results in a decrease in the $\bar{r}$ value that is detected by the chart. The chart clearly shows that process C is out of control.

- The points in region D were sampled from a circular normal population with $\sigma_x = 1$ except that the center of the distribution was shifted from the target at $(x, y) = (0, 0)$ to $(2.33, 0)$. (Any new center 2.33 units from the origin would generate similar data. The reason that a displacement of 2.33 units was used will be explained shortly.) Even though the scatter in the (x, y) points for region D is the same as the data in region A, most of the points fall farther from the target which drives the $\bar{r}$ value up again. The chart clearly shows that process D is out of control.

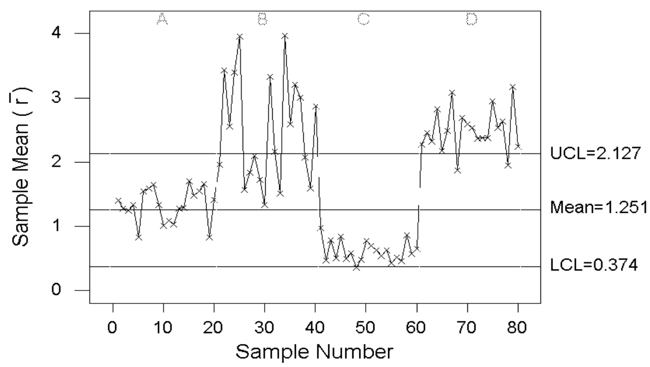

Figure 14: Example $\bar{r}$ Chart for Circular Normal Process Showing Out-of-Control Points

Regions B and D in Figure 14 are both out of control on the high side of the control chart. Close examination shows that the two data sets have the same grand mean $\bar{\bar{r}}$, however, the chart does not show that B and D are out of control for different reasons. (The data in B and D were specially constructed to have the same mean. B is centered correctly and has $\sigma_x = 2$. The center of D is displaced from the origin by 2.33 units and has $\sigma_x = 1$. These choices give both B and D $\mu_r = 2.51$.) This demonstrates one of the weaknesses of the circular normal control chart for $\bar{r}$ - it does not distinguish between the two ways that a process can go out of control. When a circular normal chart for $\bar{r}$ goes out of control it is necessary to use other techniques to determine the cause. The cause, either a change in σ_x or a displacement of the distribution center from the target, must be determined before the appropriate remedial action can be determined. If the data are (x, y) instead of just r, plotting the (x, y) or control charting both x and y can identify the problem. However, when the data are just the r values another approach must be considered. The correct analysis is to construct the circular normal plot of the r values (not the $\bar{r}$ values).

Reconsider the data from regions B and D in Figure 14. Upon initial inspection you might expect the data from these regions to give similar circular normal plots, but they do not. Both data sets are shown in the circular normal probability plot in Figure 15. Since the two data sets have the same grand mean the single line shown in the Figure applies to both of them. The data from region B fall nicely along this line indicating that these data are circular normal, however, the data from region D do not fall along this line indicating that these data are not circular normal about the intended target $(0, 0)$. Larger shifts from the target position than in D will cause points on the $\bar{r}$ chart to go farther out of control on the high side and the discrepancy will be more apparent on the circular normal probability plot. When shifts from the target are very large the r data will actually approach a normal distribution which can be checked with a normal probability plot. The normal plot of the data from region D is shown in Figure 16. This model (normal) describes the D data much better than does the circular normal model. If the data from region D were mean adjusted the transformed D values would be circular normal again.

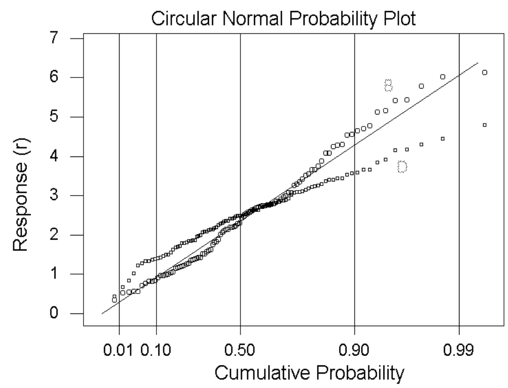

Figure 15: Centered and Shifted Circular Normal Data

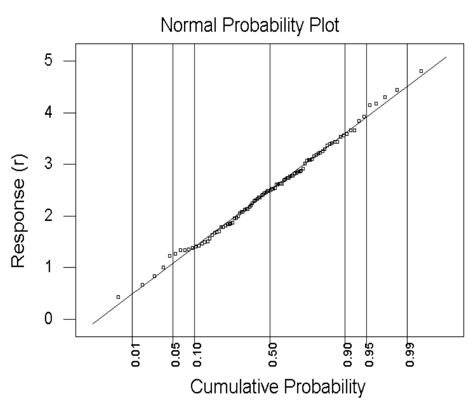

Figure 16: Normal Probability Plot of Data from Region D

1.14 Process Capability

While it is possible to define process capability statistics c_p and c_{pk} for circular normal data, they would require special interpretation compared to the usual interpretation of these values. Since c_p and c_{pk} values are so frequently misused anyway, this just provides more opportunity for confusion. Rather, it is recommended that process capability for a circular normal process be communicated in terms of the fraction defective. For a circular normal process with σ_x and upper spec limit r_{USL} Equation 25 gives the appropriate fraction defective. Suppose that we have a process that is known to have $\sigma_x = 1$ and $r_{USL} = 4.5$. The process will deliver a fraction defective given by:

$$\begin{aligned} p &= e^{-\frac{1}{2}\left(\frac{r_{USL}}{\sigma_x}\right)^2} \\ &= e^{-\frac{1}{2}\left(\frac{4.5}{1}\right)^2} \\ &= 0.000040 \\ &= 40dpm \end{aligned} \tag{43}$$

By comparison a normally distributed quality characteristic with spec $USL/LSL = \mu \pm 6\sigma$ and a 1.5σ shift in the process mean (so the process mean is just 4.5σ from a spec limit) delivers a defective rate of $3.5dpm$. This just demonstrates again that the circular normal distribution has an exaggerated right tail compared to the normal distribution, and generally upper specification limits for circular normal processes will have to be set to higher values than for normal processes to obtain the same low defective rates.