Propagation of Error

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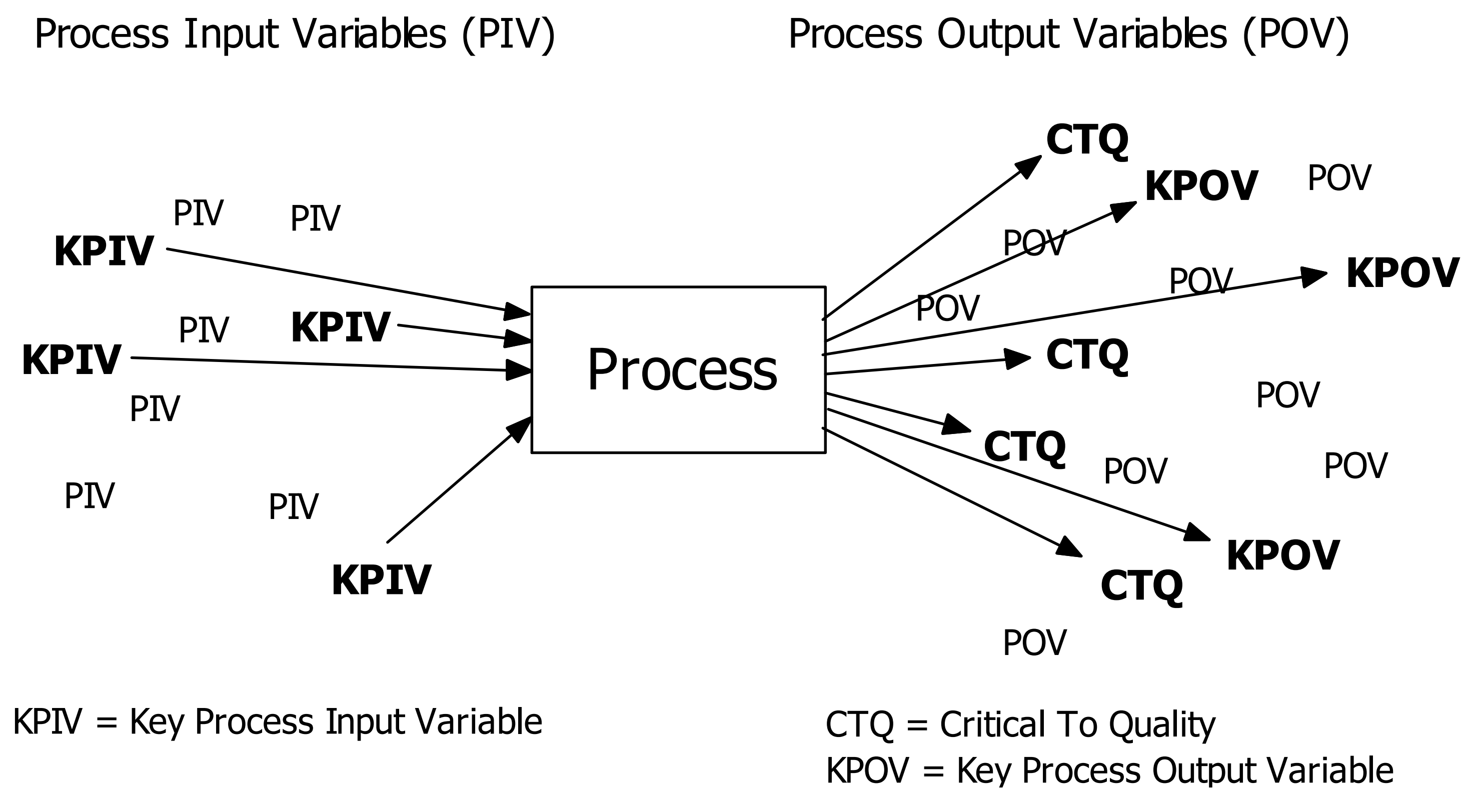
PIVs and POVs

The output variables of a process (POV) depend on its process input variables (PIV):

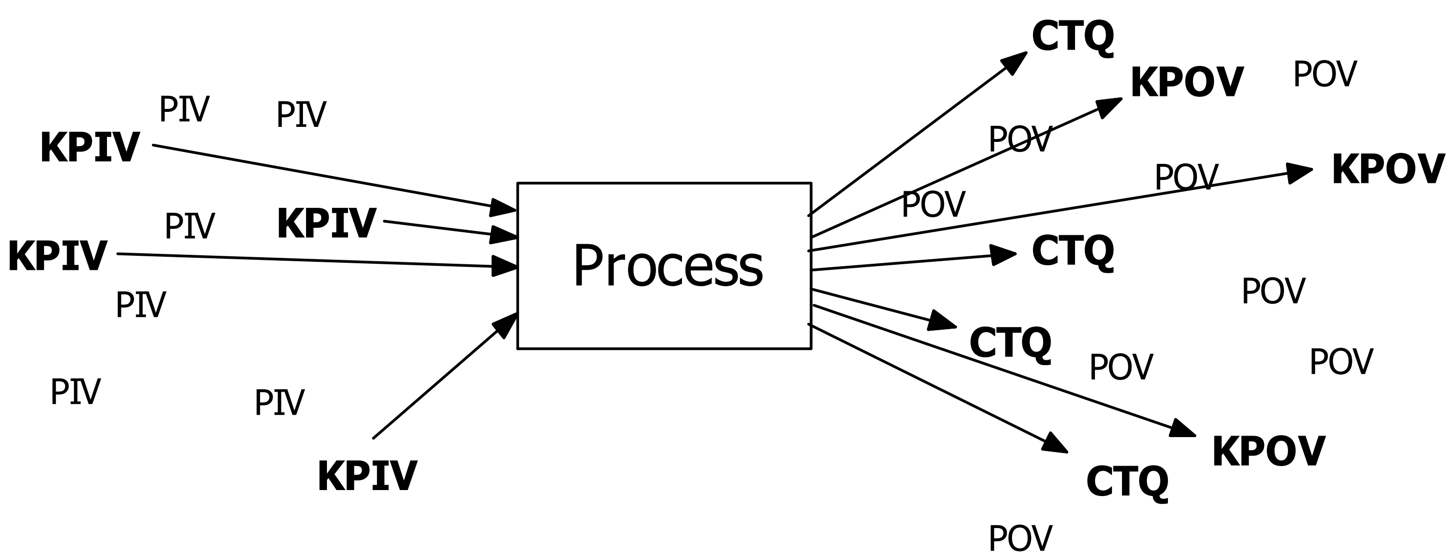


Propagation of Error

When the response (*CTQ* or *KPOV*) is a function of one or more input variables (*KPIV*s), variation in the input variables induces variation in the response. This effect is known as *propagation of error*.

Process Input Variables (PIV)

Process Output Variables (POV)



KPIV = Key Process Input Variable

CTQ = Critical To Quality
KPOV = Key Process Output Variable

Linear Stackup

In the case of a simple linear stackup, the mean y is just the sum of the x means:

$$\mu_y = \mu_{x_1} + \mu_{x_2} + \cdots$$

and the y variance is just the sum of the x variances:

$$\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \cdots + \sigma_\epsilon^2.$$

Note: In what follows, I should be explicitly including the σ_ϵ^2 term's contribution to σ_y^2 but I've been lazy. That will be important when σ_ϵ^2 is a big contributor to σ_y^2.

Weighting Factors

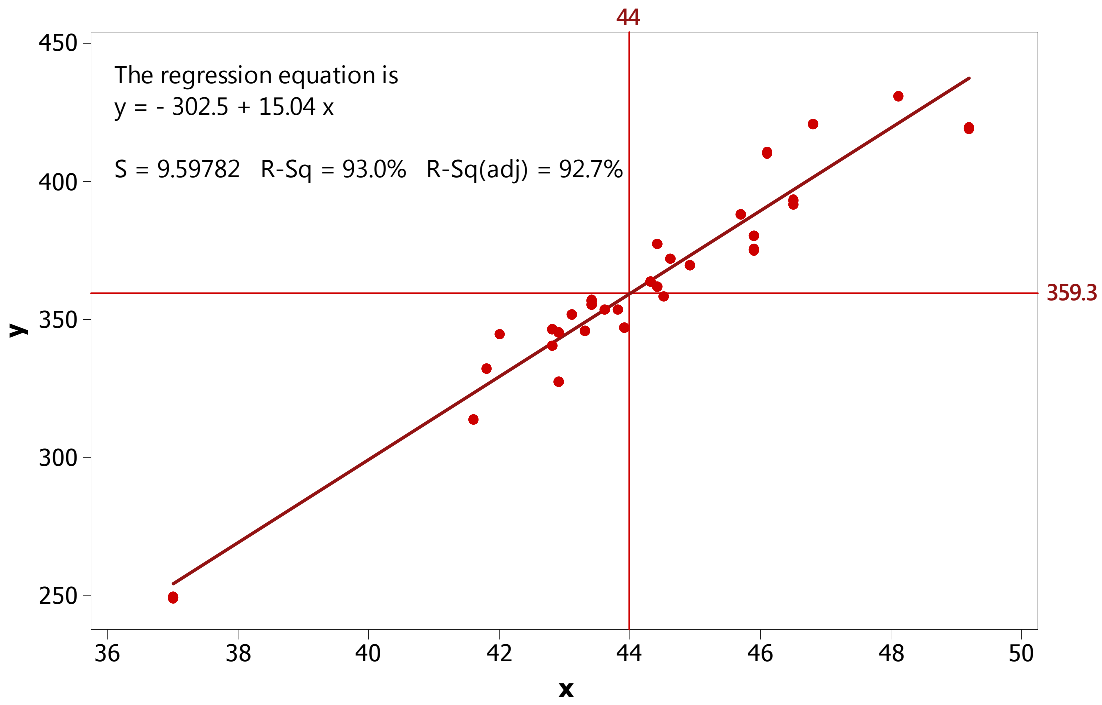
If the sensitivity of y to the xs is variable, then those sensitivities need to be taken into account:

$$\sigma_y^2 = (w_1\sigma_{x_1})^2 + (w_2\sigma_{x_2})^2 + \cdots + \sigma_\epsilon^2.$$

We need to figure out how to determine the w_i.

Propagation of Error

If we could hold x constant then the only contribution to σ_y^2 would come from the error σ_ϵ^2.



Propagation of Error

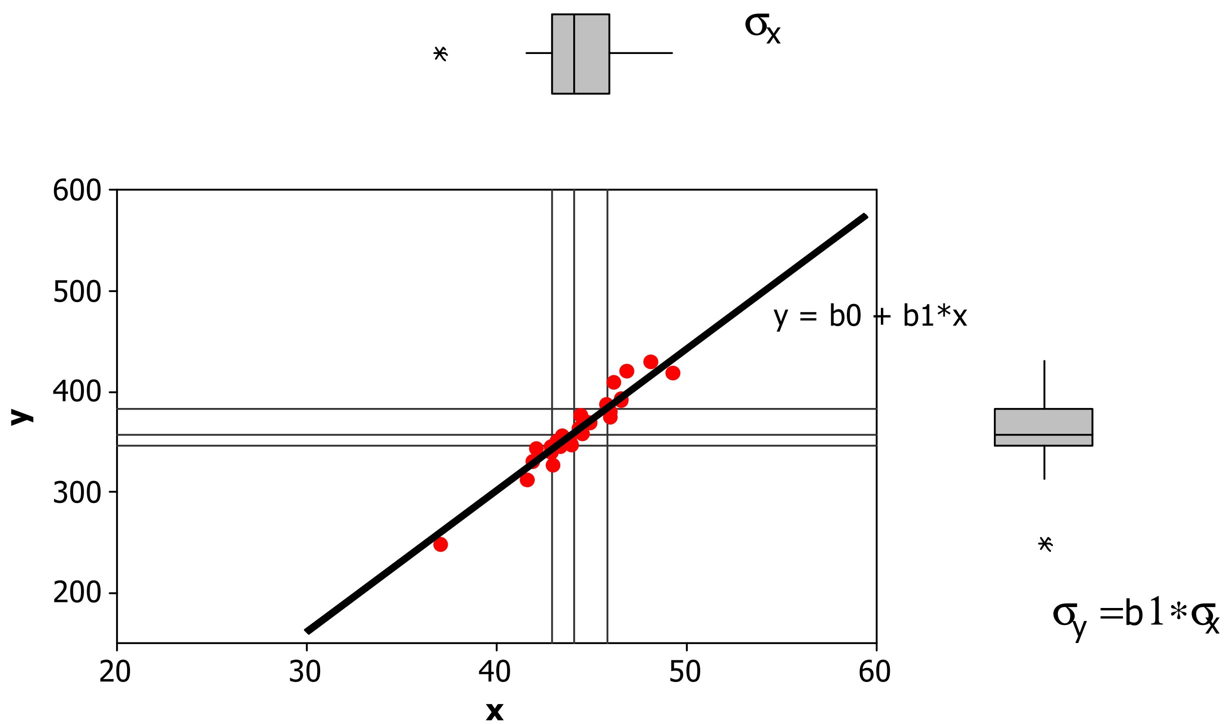
- When the transfer function that relates the response (y) to the input variables (x_i) is known, and when process capability studies have been used to quantify the variation in the input variables (σ_{x_i}), the propagated variation in the response (σ_y) can be calculated.
- If the propagated variation in the response is excessive, the individual contributions from the input variables to the total variation can be compared to prioritize those variables for future improvement opportunities.
- The transfer fuction $y = f(x_1, x_{2,\ldots})$ can be:
  - a theoretical equation from first physical principles
  - an empirical function from a designed experiment
  - a black box model such as expressed in a complicated Excel spreadsheet, finite element analysis, etc.

If y depends on a single x according to:

$$y = \beta_0 + \beta_1 x$$

then variation in x quantified by σ_x will cause or *propagate* variation in y according to:

$$\sigma_y = \beta_1 \sigma_x$$



- Standard deviations are not additive but variances are, so if y depends on two (or more) variables x_1 and x_2 according to:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

which are independent of each other, then variation in x_1 and x_2 will propagate variation to y according to:

$$\sigma_y^2 = (\beta_1 \sigma_{x_1})^2 + (\beta_2 \sigma_{x_2})^2$$

- If x_1 and x_2 are not independent of each other then:

$$\sigma_y^2 = (\beta_1 \sigma_{x_1})^2 + (\beta_2 \sigma_{x_2})^2 + 2\beta_1 \beta_2 r_{12} \sigma_{x_1} \sigma_{x_2}$$

where r_{12} is the correlation coefficient between x_1 and x_2.
- If y is a complicated function of the x_i, then in general:

$$\sigma_y^2 = \sum_i \sum_j \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} r_{ij} \sigma_{x_i} \sigma_{x_j}$$

where the partial derivatives are evaluated at the point of interest in the design space. When x_i and x_j are independent, then $r_{ij} = 0$.

Propagation of Error - Example

As an example, suppose that the transfer function for y is given by:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2$$

and that x_1 and x_2 are independent of each other. Then:

$$\frac{\partial y}{\partial x_1} = \beta_1 + \beta_{12} x_2 + 2\beta_{11} x_1$$

$$\frac{\partial y}{\partial x_2} = \beta_2 + \beta_{12} x_1$$

and the propagated error is:

$$\begin{aligned} \sigma_y^2 &= \left(\frac{\partial y}{\partial x_1} \sigma_{x_1} \right)^2 + \left(\frac{\partial y}{\partial x_2} \sigma_{x_2} \right)^2 \\ &= ((\beta_1 + \beta_{12} x_2 + 2\beta_{11} x_1)\sigma_{x_1})^2 + ((\beta_2 + \beta_{12} x_1)\sigma_{x_2})^2 \end{aligned}$$

Propagation of Error - Example

Note that this result

$$\sigma_y^2 = ((\beta_1 + \beta_{12}x_2 + 2\beta_{11}x_1)\sigma_{x_1})^2 + ((\beta_2 + \beta_{12}x_1)\sigma_{x_2})^2$$

offers multiple strategies for reducing σ_y^2:

- Reduce σ_{x_1}
- Reduce σ_{x_2}
- Choose x_1 and x_2 to make $\beta_1 + \beta_{12}x_2 + 2\beta_{11}x_1$ small, near 0
- Choose x_1 to make $\beta_2 + \beta_{12}x_1$ small, near 0

Propagation of Error - Example

Calculate the variance participation factors (VPF), i.e. the fractions of the total variance caused by each contribution:

$$VPF_1 = \frac{\left(\frac{\partial y}{\partial x_1}\sigma_{x_1}\right)^2}{\sigma_y^2}$$

$$VPF_2 = \frac{\left(\frac{\partial y}{\partial x_2}\sigma_{x_2}\right)^2}{\sigma_y^2}$$

Use the VPFs to determine which variables deserve action and which can be ignored or even relaxed.

Propagation of Error - Example

- Use the transfer function for y to optimize the response, i.e. either 1) maximize the response, 2) minimize the response, or 3) set the response to a target value, by manipulating x_1 and x_2:

$$\begin{aligned}\mu_y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 \\ &= f(x_1, x_2)\end{aligned}$$

- Use the transfer function for σ_y to identify opportunities to reduce σ_y by manipulating x_1, x_2, σ_{x_1}, and σ_{x_2}:

$$\begin{aligned}\sigma_y^2 &= ((\beta_1 + \beta_{12} x_2 + 2\beta_{11} x_1)\sigma_{x_1})^2 + ((\beta_2 + \beta_{12} x_1)\sigma_{x_2})^2 \\ &= g(x_1, x_2, \sigma_{x_1}, \sigma_{x_2})\end{aligned}$$

- Simultaneously optimize μ_y while minimizing σ_y or ...

Propagation of Error - Example

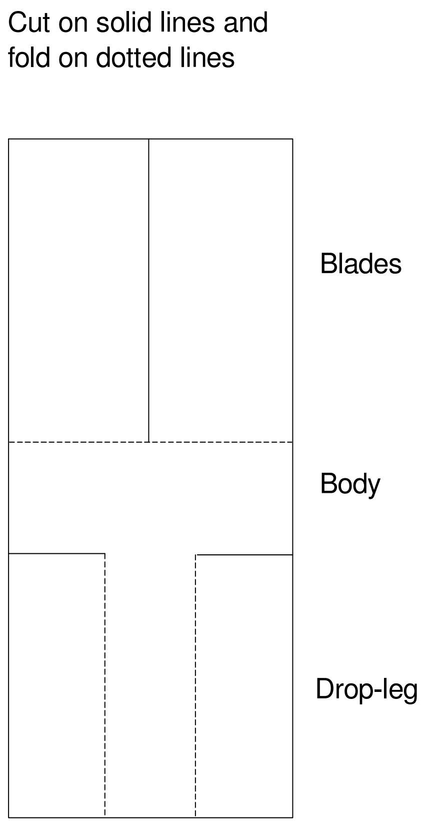
Tip: Instead of struggling to simultaneously optimize $\mu_y = f(x_1, x_2)$ while minimizing $\sigma_y^2 = g(x_1, x_2, \sigma_{x_1}, \sigma_{x_2})$ GE uses μ_y and σ_y to calculate the total defective rate relative to specification limits on y:

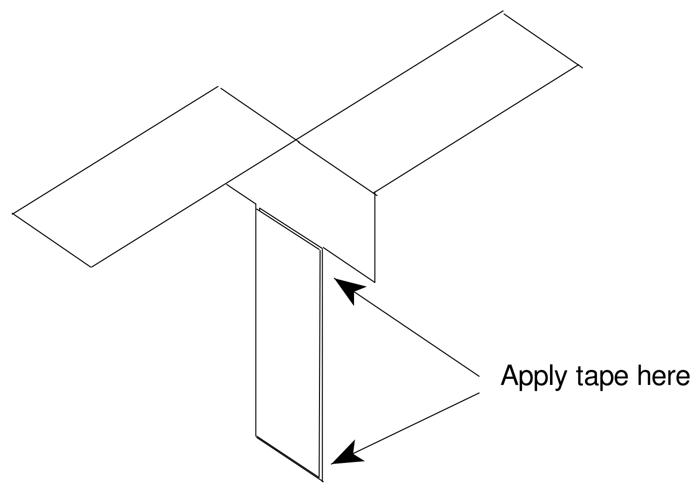
$$p_{total} = 1 - \Phi(LSL < y < USL; \mu_y, \sigma_y)$$

and then minimizes

$$p_{total} = h_1(x_1, x_2).$$

Example: Use your solution to DOE homework Problem 9.2 to a) maximize paper helicopter flight time and b) determine the induced variation in flight time due to 10% (one sigma) manufacturing variation in blade width and length.





Solution: The experiment's variables matrix was:

Variable	-1	$+1$	*Units*
A : *Width*	1.25	2	*inch*
B : *Length*	2	4	*inch*
C : *Folds*	1	2	*NA*

The equation for flight time in coded units was determined to be:

$$Time = 3.93 - 0.22A + 1.03B - 0.15AB$$

where A is the blade width and B is the blade length.

The flight time is maximized for $A = -1\ (1.25in)$ and $B = 1\ (4.0in)$.
Under these conditions the predicted flight time is:

$$\begin{aligned} Time &= 3.93 - 0.22(-1) + 1.03(1) - 0.15(-1)(1) \\ &= 3.93 + 0.22 + 1.03 + 0.15 \\ &= 5.33s \end{aligned}$$

For the optimal helicopter with 4 inch long blades, if the one sigma manufacturing variation in length is 10% or $0.10 \times 4.0in = 0.4in$ for this helicopter, then the corresponding standard deviation in coded units is:

$$\sigma_B = 0.4in\left(\frac{2}{2in}\right) = 0.4$$

Likewise, For the optimal helicopter with 1.25 inch wide blades, if the one sigma manufacturing variation in width is 10% or $0.10 \times 1.25in = 0.125in$ for this helicopter, then the corresponding standard deviation in coded units is:

$$\sigma_A = 0.125in\left(\frac{2}{0.75in}\right) = 0.333$$

The partial derivatives evaluated at the nominal conditions are:

$$\left.\frac{\partial Time}{\partial A}\right|_{nominal} = \left.\frac{\partial}{\partial A}(3.93 - 0.22A + 1.03B - 0.15AB)\right|_{nominal}$$

$$= -0.22 - 0.15B|_{B=1}$$

$$= -0.37$$

and

$$\left.\frac{\partial Time}{\partial B}\right|_{nominal} = \left.\frac{\partial}{\partial B}(3.93 - 0.22A + 1.03B - 0.15AB)\right|_{nominal}$$

$$= 1.03 - 0.15A|_{A=-1}$$

$$= 1.18$$

PIV	Nom	σ	$\left.\frac{\partial Comp}{\partial PIV_i}\right\|_{Nom}$	$\left(\left.\frac{\partial Comp}{\partial PIV_i}\right\|_{Nom}\sigma_{PIV_i}\right)^2$	VPF
A	-1	0.333	-0.37	0.0152	0.064
B	1	0.400	1.18	0.223	0.936
			$Total$	0.238	1.000

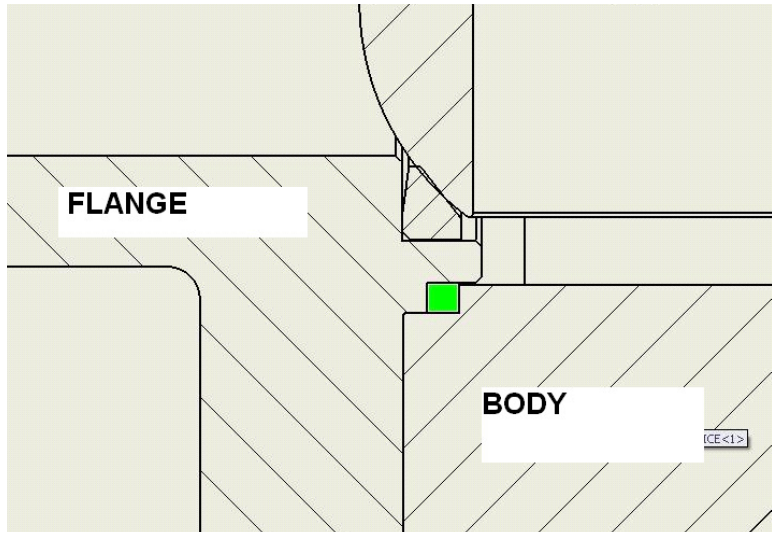
So the total variation propagated to the flight time response from blade width and blade length variation is

$$\sigma_{Time} = \sqrt{0.238}$$
$$= 0.488 \text{ seconds}$$

That is, about one half second of variation in flight time will be induced by 10% one-sigma variation in the blade width and length. 93.6% of the variation in flight time will be caused by variation in the blade length, so any process improvement efforts should be directed at tightening the tolerances on the blade length.

Example: Seal compression is given by

$$Compression = \frac{V_{Seal}}{V_{Gland}} - 1$$



V_{Seal} is the volume of the seal:

$$V_{Seal} = \frac{\pi}{4} ST(SOD^2 - SID^2)$$

V_{Gland} is the volume of the gland:

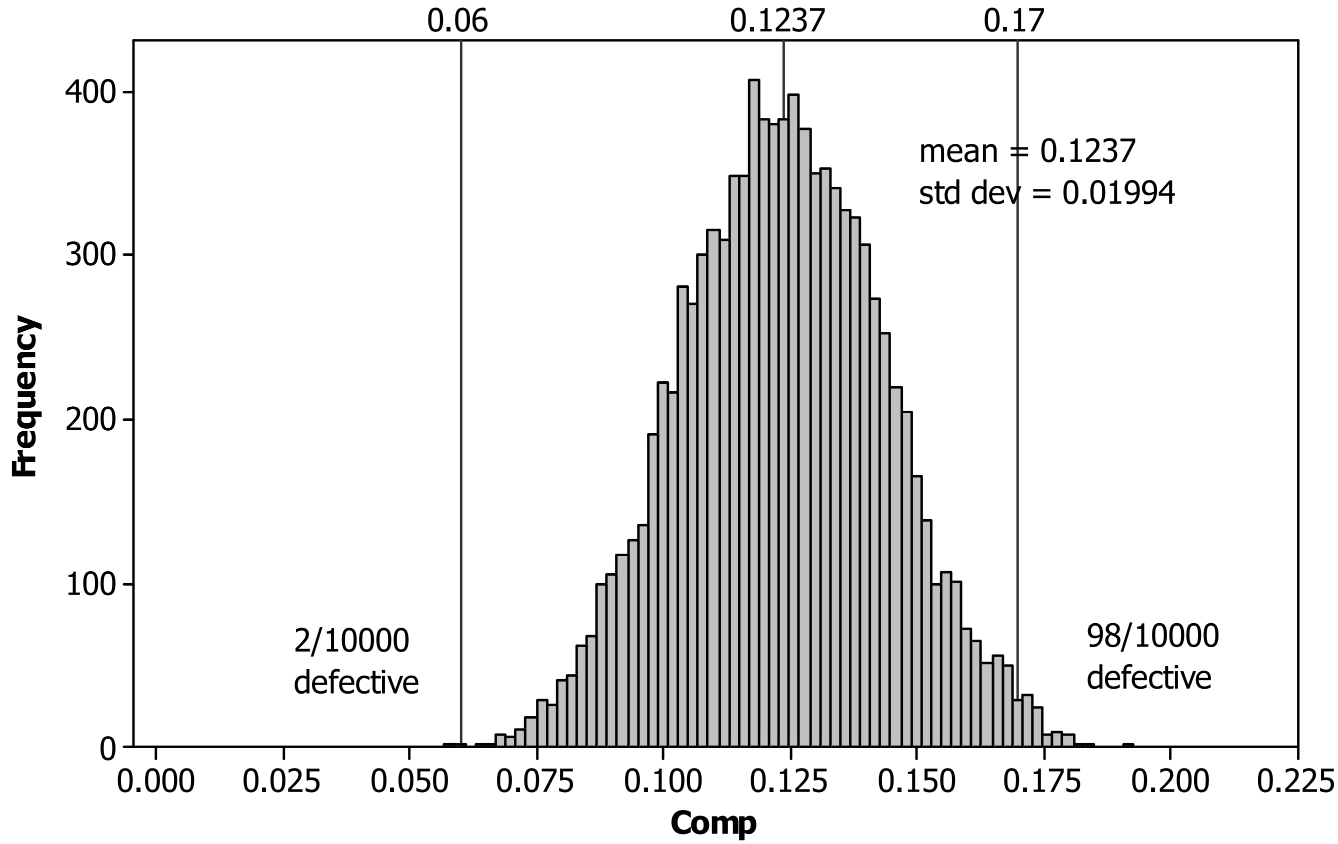
$$V_{Gland} = \frac{\pi}{4}(BCD - FSH)(BSD^2 - FBD^2)$$

Characteristic	Abbreviation	Min	Nom	Max
Seal Outside Diameter	SOD	2.193	2.195	2.197
Seal Inside Diameter	SID	2.002	2.004	2.006
Seal Thickness	ST	0.122	0.124	0.126
Body Seal Diameter	BSD	2.197	2.198	2.199
Body Counterbore Depth	BCD	0.175	0.176	0.177
Flange Step Height	FSH	0.067	0.069	0.071
Flange Boss Diameter	FBD	2.000	2.001	2.002

- The minimum specification limit on compression is 6%. If the compression is significantly lower the seal leaks.
- The nominal compression is 12.4%.
- The worst case compression is 4.2% but how often does that happen?
- What is the effect of the component tolerances on compression?

Solution By Simulation:

Simulation of 10000 seal assemblies assuming uniform distributions:



```
random 10000 c1;
   uniform 2.193 2.197.
rand 10000 c2;
   uniform 2.002 2.006.
rand 10000 c3;
   uniform 0.122 0.126.
rand 10000 c4;
   uniform 2.197 2.199.
rand 10000 c5;
   uniform 0.175 0.177.
rand 10000 c6;
   uniform 0.067 0.071.
rand 10000 c7;
   uniform 2.0 2.002.
let c8 = 3.14/4 * c3 *(c1**2 - c2**2)
let c9 = 3.14/4 * (c5 - c6) * (c4**2 - c7**2)
let c10 = c8 / c9 - 1
```

Analytical Method

The simulation method doesn't quantify the individual contributions to the overall variability in compression; however, by the analytical method:

$$\sigma_{Comp} = \sqrt{\left(\frac{\partial Comp}{\partial SID}\sigma_{SID}\right)^2 + \cdots + \left(\frac{\partial Comp}{\partial FBD}\sigma_{FBD}\right)^2}$$

For a uniform distribution with specification limits USL and LSL:

$$\mu = \frac{USL + LSL}{2}$$

$$\sigma = \frac{USL - LSL}{\sqrt{12}}$$

The partial derivatives are evaluated at the nominal dimensions. For example:

$$\left.\frac{\partial Comp}{\partial SID}\right|_{nominal} = \left.\frac{\partial}{\partial SID}\left(\frac{ST(SOD^2 - SID^2)}{(BCD - FSH)(BSD^2 - FBD^2)}\right)\right|_{nominal}$$

$$= \left.\frac{-2 \times ST \times SID}{(BCD - FSH)(BSD^2 - FBD^2)}\right|_{nominal}$$

$$= \frac{-2 \times 0.124 \times 2.004}{(0.176 - 0.069)(2.198^2 - 2.001^2)}$$

$$= -5.615$$

Evaluating all of the contributions:

PIV	Nom	$Max - Min$	σ_{PIV}	$\frac{\partial Comp}{\partial PIV_i}$	$\left(\frac{\partial Comp}{\partial PIV_i}\sigma_{PIV_i}\right)^2$	VPF
SOD	2.195	0.004	0.001155	6.150	0.000050	0.124
SID	2.004	0.004	0.001155	–5.615	0.000042	0.103
ST	0.124	0.004	0.001155	9.061	0.000110	0.269
BSD	2.198	0.002	0.000577	–5.971	0.000012	0.029
BCD	0.176	0.002	0.000577	–10.50	0.000037	0.090
FSH	0.069	0.004	0.001155	10.50	0.000147	0.361
FBD	2.001	0.002	0.000577	5.436	0.000010	0.024
				$Total$	0.000408	1.000

- So the total variation propagated to the compression from the seven process input variables is

$$\sigma_{Comp} = \sqrt{0.000408}$$
$$= 0.0202$$

which is in excellent agreement with the simulation.

- The largest contributors to variation in compression are flange step height (FSH) and seal thickness (ST) which account for 63% of the total variation.
- Note that FSH and ST act in the same horizontal direction in the cross section view above.

Further Questions

- Is it reasonable to ignore all variation in component dimensions except flange step height and seal thickness?
- What if the component distributions are normal, centered in their specs, with $USL - LSL = 4\sigma$?
- What if the component distributions are normal, biased by 1.5σ in directions to reduce compression, with $USL - LSL = 4\sigma$?

