Binary Logistic Regression

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Experiments

- The purpose of an experiment is to determine how a response (or dependent variable) changes with respect to one or more independent variables (or knobs). This relationship is expressed as a mathematical predictive model of the form:

$$y = f(x_1, x_2, \cdots)$$

- Variables, whether they are independent or dependent, can be qualitative (attribute) or quantitative (measurement) in nature.
- The qualitative/quantitative nature of the independent and dependent variables in an experiment determine the method of statistical analysis.

Types of Experimental Responses

- Variables (or Measurement or Quantitative) Responses
  - Measurement scale is interval or ratio
  - Examples:
    - Length
    - Time
    - Weight or mass
    - Temperature
- Attribute (or Qualitative) Responses
  - Binary Responses - Two possible outcome states, e.g. pass/fail, go/no-go, dead/alive, etc.
  - Ordinal Responses - Observations can be arranged by size but the measurement scale is ordinal, not interval or ratio.
  - Nominal Responses - Observations are sorted into three or more qualitative categories.

Analysis of Binary Responses

The usual methods of analysis for quantitative responses (regression and ANOVA) are not appropriate for binary responses because:

- The distribution of the residuals is not normal.
- The variance of the residuals is not constant, i.e. the residuals in the model for a binary response are heteroscedastic.
- The response is constrained to the range $[0,1]$.

These problems necessitate the use of an alternative method of model fitting for binary responses called *maximum likelihood*. In the maximum likelihood method, the model coefficients are those most likely to produce the observed experimental results.

Binary Response Example: Toxicity Study

Suppose that a pharmaceutical company wants to evaluate the toxicity of an experimental drug. An appropriate experiment would expose animals to different dosages of the drug.

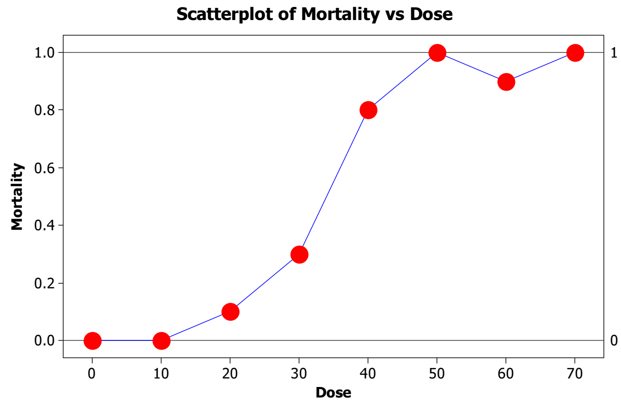
Dose	n	Died	Survived	Mortality
0	10	0	10	0
10	10	0	10	0
20	10	1	9	0.1
30	10	3	7	0.3
40	10	8	2	0.8
50	10	10	0	1.0
60	10	9	1	0.9
70	10	10	0	1.0

Binary Response Example: Toxicity Study

- The response is binary - each animal falls into one of two possible categories: died or survived.
- The desired model will provide a prediction for mortality as a function of dose.
- Mortality is limited to the range $[0,1]$.
- The model will be incapable of predicting whether any given animal survives or not, but can only provide a probability of survival based on the dose that the animal received.

Binary Response Example: Toxicity Study

- The relationship between mortality and dose is inherently nonlinear.



- The usual strategy for analysis of nonlinear relationships is to apply a mathematical transformation to one or both of the variables that linearizes the relationship so that the usual linear regression methods can be used.

Transformations for Binary Responses

- Freeman-Tukey:

$$p' = sin^{-1}(\sqrt{p})$$

- Normit:

$$p' = z_p$$

- Logit:

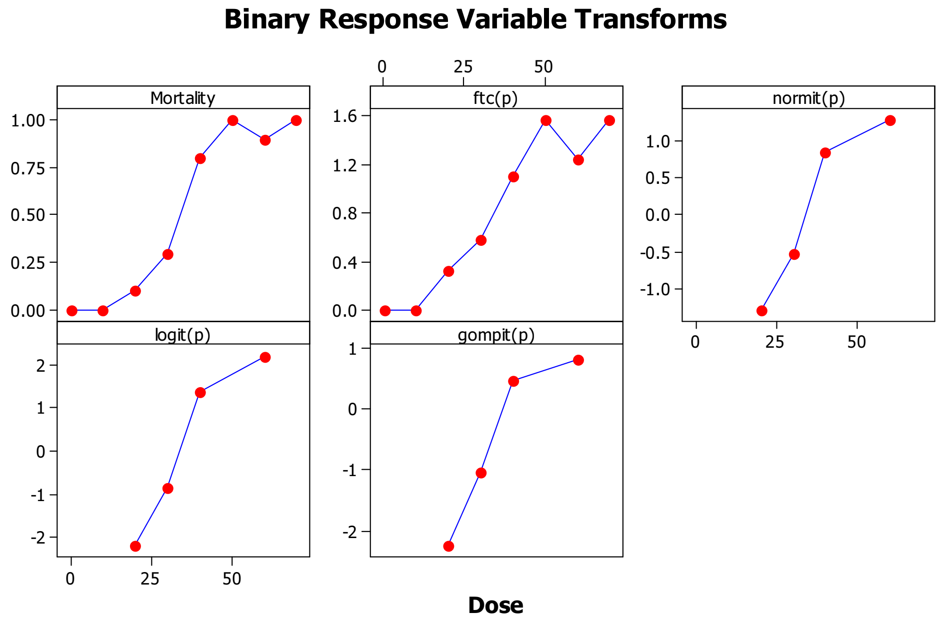
$$p' = ln\left(\frac{p}{1-p}\right)$$

- Gompit:

$$p' = ln(-ln(1-p))$$

- All of these transforms have similar effects, but one might fit the data better than the others.
- Use the appropriate transformation for the situation.
- When a first-principles model is not obvious, the *logit* transform is preferred.

Binary Response Example: Toxicity Study



Binary Response Example: Toxicity Study

Binary Logistic Regression: Died, n versus Dose

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Link Function: Logit


Response Information

Variable  Value    Count
Died      Success     41
          Failure     39
n         Total       80


Logistic Regression Table

                                                Odds     95% CI
Predictor      Coef    SE Coef      Z      P   Ratio  Lower  Upper
Constant   -5.57928    1.33976  -4.16  0.000
Dose       0.164110  0.0375423   4.37  0.000    1.18   1.09   1.27

OR = exp(0.164) = 1.18  => Odds of death increase by 18% per gram of dose

Log-Likelihood = -19.909
Test that all slopes are zero: G = 71.035, DF = 1, P-Value = 0.000


Goodness-of-Fit Tests

Method           Chi-Square  DF      P
Pearson             6.77692   6  0.342
Deviance            4.58968   6  0.597
Hosmer-Lemeshow     6.77692   6  0.342
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Binary Response Example: Toxicity Study

Table of Observed and Expected Frequencies:
(See Hosmer-Lemeshow Test for the Pearson Chi-Square Statistic)

Value	Group 1	2	3	4	5	6	7	8	Total
Success									
Obs	0	0	1	3	8	10	9	10	41
Exp	0.0	0.2	0.9	3.4	7.3	9.3	9.9	10.0	
Failure									
Obs	10	10	9	7	2	0	1	0	39
Exp	10.0	9.8	9.1	6.6	2.7	0.7	0.1	0.0	
Total	10	10	10	10	10	10	10	10	80

Measures of Association:
(Between the Response Variable and Predicted Probabilities)

Pairs	Number	Percent	Summary Measures	
Concordant	1507	94.2	Somers' D	0.92
Discordant	37	2.3	Goodman-Kruskal Gamma	0.95
Ties	55	3.4	Kendall's Tau-a	0.47
Total	1599	100.0		

Predictions

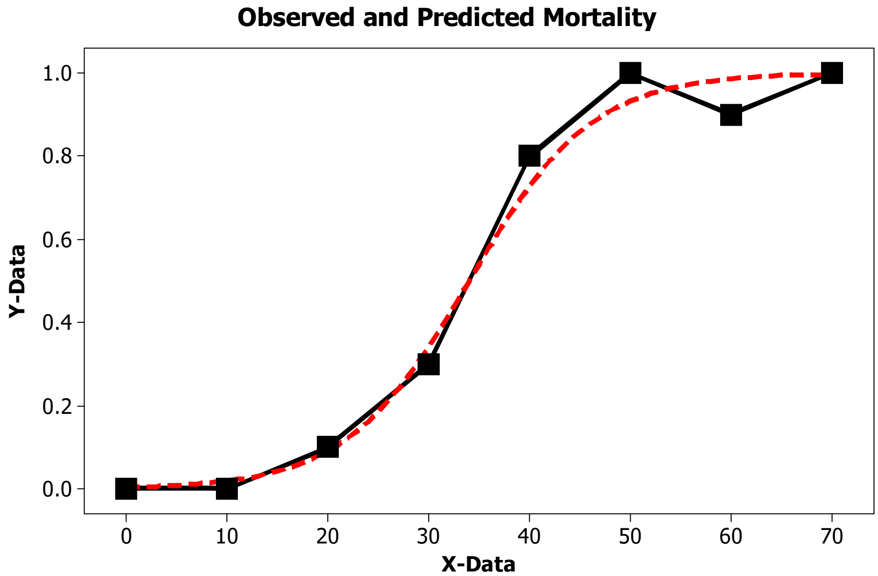
The binary logistic regression model has the form:

$$y_i = \ln\left(\frac{p_i}{1-p_i}\right) = b_0 + b_1 x_i + \epsilon_i$$

This equation can be solved for p_i:

$$\widehat{p}_i = \frac{e^{\widehat{y}_i}}{1+e^{\widehat{y}_i}} = \frac{e^{b_0+b_1x_i}}{1+e^{b_0+b_1x_i}}$$

Binary Response Example: Toxicity Study



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Binary Response Example: Toxicity Study

Predictions:

- LD50: 50% of the animals will die when:

$$ln\left(\frac{0.5}{0.5}\right) = 0 = -5.58 + 0.164 \times Dose$$

$$Dose = \frac{5.58}{0.164} = 34$$

- Mortality at Dose = 10?

$$ln\left(\frac{p}{1-p}\right) = -5.58 + 0.164 \times 10 = -3.94$$

$$p = \frac{e^{-3.94}}{1 + e^{-3.94}} = 0.019$$

Binary Logistic Regression Model Diagnostics

Must still validate the model by checking for:

- Outliers
- Lack of Fit
- Leverage

Challenger

The data below are the number of SRM o-ring failures (f) and temperatures (T) at launch time prior to the Challenger accident. Was there sufficient evidence to suspend the launch? The temperature on the day of the launch was 31F.

ID	T	n	f	ID	T	n	f
1	66	6	0	13	70	6	1
2	70	6	1	14	67	6	0
3	69	6	0	15	53	6	2
4	80	6	0	16	75	6	0
5	68	6	0	17	67	6	0
6	67	6	0	18	70	6	0
7	72	6	0	19	81	6	0
8	73	6	0	20	76	6	0
9	70	6	0	21	79	6	0
10	57	6	1	22	75	6	0
11	63	6	1	23	76	6	0
12	78	6	0	24	58	6	1

Challenger

Binary Logistic Regression: f, n versus T

Variable	Value	Count
f	Event	7
	Non-event	137
n	Total	144

Logistic Regression Table

					Odds	95% CI	
Predictor	Coef	SE	Z	P	Ratio	Lower	Upper
Constant	8.90	3.579	2.49	0.013			
T	-0.181	0.058	-3.13	0.002	0.83	0.75	0.93

Log-Likelihood = -22.123

Test that all slopes are zero:

G = 11.744, DF = 1, P-Value = 0.001

Challenger

Goodness-of-Fit Tests

Method	Chi-Square	DF	P
Pearson	6.57752	15	0.968
Deviance	6.61859	15	0.967
Hosmer-Lemeshow	4.48632	5	0.482

Measures of Association:

(Between the Response Variable and Predicted Probabilities)

Pairs	Number	Percent	Summary Measures	
Concordant	755	78.7	Somers' D	0.64
Discordant	137	14.3	Goodman-Kruskal Gamma	0.69
Ties	67	7.0	Kendall's Tau-a	0.06
Total	959	100.0		

Challenger

The model is:

$$\ln\left(\frac{p}{1-p}\right) = 8.9 - 0.181 \times T.$$

At the time of the launch $T = 31$, so:

$$\ln\left(\frac{p}{1-p}\right) = 8.9 - 0.181 \times 31 = 3.29$$

and

$$p = \frac{e^{3.29}}{1 + e^{3.29}} = 0.964.$$

That is, the model predicts that on the day of the launch the probability that an o-ring would fail was 96.4%. The probability that there would be no o-ring failures among the six o-rings on the flight was:

$$b(x = 0; n = 6, p = 0.964) = 0.$$

References

- Neter, Kutner, Nachtsheim, and Wasserman (1996), *Applied Linear Statistical Models*, 4th Ed., McGraw-Hill.
- Agresti (2002), *Categorical Data Analysis*, 2nd Ed., Wiley.
- Christensen (1997), *Log-Linear Models and Logistic Regression*, 2nd Ed., Springer.
- Hosmer and Lemeshow, *Applied Logistic Regression*, Wiley, 1989.